Third Part: Substructural Logics

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Why proof theory?

While algebra focuses primarily on structures and their properties, logic concerns itself more with syntax and deduction. Despite these differences, algebraization provides a powerful match between syntactically defined axiom systems and classes of algebras. In particular, theorems of classical or intuitionistic logic can be translated into equations holding in all Boolean or Heyting algebras and vice versa. To the algebraist, this may suggest that propositional logic is just "algebra in disguise."
A logician may respond that:

1. some logics are not algebraizable;
2. the case for an algebraic approach to first order logic is not so compelling;
3. syntactic presentations offer an alternative perspective that pure semantics cannot provide. In particular, syntactic objects such as formulas, equations, and proofs, may be investigated themselves as first-class citizens using methods such as induction on formula complexity or height of a proof.

Gentzen's approach

A limitation of the early period of proof theory (the Hilbert school, 1920s) was the reliance on axiomatizations typically consisting of many axiom schemata and just a few rules. The axiomatic approach is flexible but does not seem to reflect the way that mathematicians, or humans in general, construct and reason about proofs, and suffers from a lack of control over proofs as mathematical objects.
These issues were addressed by Gerhard Gentzen (1935) via the introduction of two new proof formalisms: *natural deduction* and the *sequent calculus*. In particular, he defined sequent calculi, GCL and GIL, for first order classical logic and first order intuitionistic logic, respectively, giving birth to an area known now as *structural proof theory*. The propositional parts of Gentzen's systems correspond directly to Boolean algebras and Heyting algebras.

Substructural logics, which themselves correspond to classes of residuated lattices, are then obtained, very roughly speaking, by removing certain rules from these systems.

Sequent calculi: an overview

A *sequent* is an ordered pair of finite sequences of formulas, written:

$$\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m.$$

Intuitively: the disjunction $\beta_1 \vee \ldots \vee \beta_m$ "follows from" the conjunction $\alpha_1 \wedge \ldots \wedge \alpha_n$.

Sequent rules are typically written schematically using $\Gamma, \Delta, \ldots$ to stand for arbitrary sequences of formulas, comma for concatenation, and an empty space for the empty sequence. They consist of *instances* with a finite set of premises and a single conclusion, rules with no premises being called *initial sequents*.

A *sequent calculus* GL is a set of sequent rules. A *derivation* in GL of a sequent S from a set of sequents X is a finite tree of sequents with root S such that each sequent is either a leaf and a member of X, or S is the conclusion and its children (if any) are the premises of an instance of a rule of the system. When such a tree exists, we say that S is derivable from X in GL and write $X \vdash_{\mathrm{GL}} S$.

The calculus GCL

$$(\text{ID})\ \frac{}{\alpha \Rightarrow \alpha} \qquad (\text{CUT})\ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\text{WL})\ \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad (\text{WR})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$$

$$(\text{CL})\ \frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad (\text{CR})\ \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}$$

$$(\text{EL})\ \frac{\Gamma, \alpha, \beta, \Pi \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Pi \Rightarrow \Delta} \qquad (\text{ER})\ \frac{\Gamma \Rightarrow \Pi, \alpha, \beta, \Delta}{\Gamma \Rightarrow \Pi, \beta, \alpha, \Delta}$$

The calculus GCL (continued)

$$(\to\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta, \Pi \Rightarrow \Sigma}{\alpha \to \beta, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow\to)\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}$$

$$(\wedge \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$(\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \qquad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \qquad (\vee \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}$$

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}$$

$$(0 \Rightarrow)\ \frac{}{0 \Rightarrow} \qquad (\Rightarrow 0)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, 0}$$

$$(1 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{1, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow 1)\ \frac{}{\Rightarrow 1}$$

An example of derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\,(\mathrm{ID})}{\alpha \Rightarrow \beta, \alpha}\,(\mathrm{WR})}{\Rightarrow \alpha \to \beta, \alpha}\,(\Rightarrow\to) \qquad \dfrac{}{\alpha \Rightarrow \alpha}\,(\mathrm{ID})}{(\alpha \to \beta) \to \alpha \Rightarrow \alpha, \alpha}\,(\to\Rightarrow)}{(\alpha \to \beta) \to \alpha \Rightarrow \alpha}\,(\mathrm{CR})}{\Rightarrow ((\alpha \to \beta) \to \alpha) \to \alpha}\,(\Rightarrow\to)$$

The calculus GIL

Gentzen's framework also accommodates a calculus GIL for intuitionistic logic, obtained from GCL simply by restricting sequents $\Gamma \Rightarrow \Delta$ so that Δ is allowed to contain at most one formula.
In particular, GIL has no right exchange or right contraction rules, and right weakening is confined to premises with empty succedents. Hence, for instance, the derivation of Peirce's law is blocked.

GCL and HCL (1)

Let $\Box(\alpha_1, \dots, \alpha_n)$ stand for $\alpha_1 \Box \dots \Box \alpha_n$ for $\Box \in \{\wedge, \vee\}$ where $\wedge()$ is 1 and $\vee()$ is 0. We define:

$$\begin{aligned} \tau(\alpha) &= \{\Rightarrow \alpha\}; \\ \rho(\Gamma \Rightarrow \Delta) &= \wedge\Gamma \to \vee\Delta. \end{aligned}$$

Theorem

$X \vdash_{\mathrm{GCL}} S$ if and only if $\{\rho(S') \mid S' \in X\} \vdash_{\mathrm{HCL}} \rho(S)$.

GCL and HCL (2)

Proof.

Left to right: induction on the height of a derivation in GCL.
Right to left. It is easily checked that for any axiom α of HCL, $\tau(\alpha)$ is derivable in GCL. If $\tau(\alpha)$ and $\tau(\alpha \to \beta)$ are derivable in GCL, then so is $\tau(\beta)$, using $(\textsc{cut})$ twice with the derivable sequent $\alpha, \alpha \to \beta \Rightarrow \beta$. Note also that for any sequent S: $S \vdash_{\mathrm{GCL}} \tau(\rho(S))$ and $\tau(\rho(S)) \vdash_{\mathrm{GCL}} S$.
Hence if

$$\{\rho(S') : S' \in X\} \vdash_{\mathrm{HCL}} \rho(S)$$

then

$$\{\tau(\rho(S')) : S' \in X\} \vdash_{\mathrm{GCL}} \tau(\rho(S))$$

and thus $X \vdash_{\mathrm{GCL}} S$. □

Cut elimination (1)

What are the advantages of GCL and GIL over Hilbert-style calculi? Proof search in the latter is hindered by the need to guess formulas α and $\alpha \to \beta$ as premisses when we apply modus ponens. On the other hand, in GCL and GIL we have to guess which formula α to use when applying $(\textsc{cut})$.

If we could do without the cut rule, in finding derivations we could just apply rules where formulas in the premisses are subformulas of formulas in the conclusion. Indeed, Gentzen showed that:

- $(\textsc{cut})$ is not needed for deriving sequents from empty sets of assumptions;
- there exists a cut elimination algorithm that transforms such derivations into cut-free derivations.

Cut elimination (2)

Theorem (Gentzen 1935)

Cut-elimination holds for GCL *and* GIL.

Proof.

(Sketch). Intuitively, the idea is to push applications of the cut rule upwards in derivations until they reach initial sequents and disappear. For example, suppose that we have a derivation in GIL ending

$$\dfrac{\dfrac{\vdots}{\Gamma_2 \Rightarrow \alpha} \quad \dfrac{\vdots}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\text{CUT})$$

□

Cut elimination (3)

Proof.

The cut-formula α occurs on the right in one premise, and on the left in the other. A natural strategy for eliminating this application of $(\textsc{cut})$ is to look at the derivations of these premises. If one of the premises is an instance of $(\textsc{id})$, then it must be $\alpha \Rightarrow \alpha$ and the other premise must be exactly the conclusion, derived with one fewer applications of $(\textsc{cut})$. Otherwise, we have two possibilities. The first is that one of the premises ends with an application of a rule where α is not the decomposed formula, e.g.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_2'' \Rightarrow \beta_1} \quad \dfrac{\vdots}{\Gamma_2', \beta_2, \Gamma_2''' \Rightarrow \alpha}}{\Gamma_2', \Gamma_2'', \beta_1 \to \beta_2, \Gamma_2''' \Rightarrow \alpha}\,(\to\Rightarrow) \quad \dfrac{\vdots}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \Gamma_2', \Gamma_2'', \beta_1 \to \beta_2, \Gamma_2''', \Gamma_3 \Rightarrow \Delta}\,(\textsc{cut})$$

Cut elimination (4)

Proof.

In this case, we can "push the cut upwards" in the derivation to get:

$$\dfrac{\dfrac{\vdots}{\Gamma_2'' \Rightarrow \beta_1} \qquad \dfrac{\dfrac{\vdots}{\Gamma_2', \beta_2, \Gamma_2''' \Rightarrow \alpha} \quad \dfrac{\vdots}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \Gamma_2', \beta_2, \Gamma_2''', \Gamma_3 \Rightarrow \Delta}\,(\text{CUT})}{\Gamma_1, \Gamma_2', \Gamma_2'', \beta_1 \to \beta_2, \Gamma_2''', \Gamma_3 \Rightarrow \Delta}\,(\to\Rightarrow)$$

That is, we have a derivation where the left premise in the new application of (CUT) has a shorter derivation than the application in the original derivation. □

Cut elimination (5)

Proof.

The second possibility is that the last application of a rule in both premises involves α as the decomposed formula, e.g.

$$\dfrac{\dfrac{\dfrac{\vdots}{\alpha_1, \Gamma_2 \Rightarrow \alpha_2}}{\Gamma_2 \Rightarrow \alpha_1 \to \alpha_2}\,(\Rightarrow\to) \qquad \dfrac{\dfrac{\vdots}{\Gamma_1'' \Rightarrow \alpha_1} \quad \dfrac{\vdots}{\Gamma_1', \alpha_2, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1', \Gamma_1'', \alpha_1 \to \alpha_2, \Gamma_3 \Rightarrow \Delta}\,(\to\Rightarrow)}{\Gamma_1', \Gamma_1'', \Gamma_2, \Gamma_3 \Rightarrow \Delta}\,(\mathrm{CUT})$$

□

Cut elimination (6)

Proof.

Here we rearrange our derivation in a different way: we replace the application of (CUT) with applications of (CUT) with cut-formulas α_1 and α_2:

$$\dfrac{\dfrac{\vdots}{\Gamma_1'' \Rightarrow \alpha_1} \quad \dfrac{\dfrac{\vdots}{\alpha_1, \Gamma_2 \Rightarrow \alpha_2} \quad \dfrac{\vdots}{\Gamma_1', \alpha_2, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1', \alpha_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\text{CUT})}{\Gamma_1', \Gamma_1'', \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\text{CUT})$$

We now have two applications of (CUT) but with cut-formulas of a smaller complexity than the original application. □

Cut elimination (7)

This procedure, formalized using a double induction on cut-formula complexity and the combined height of derivations of the premises, eliminates applications of (CUT) for many sequent calculi. However, it encounters a problem with contraction. Consider the following situation:

$$\dfrac{\dfrac{\vdots}{\Gamma_2 \Rightarrow \alpha} \quad \dfrac{\dfrac{\vdots}{\Gamma_1, \alpha, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}\,(\text{CL})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\,(\text{CUT})$$

In this case we need to perform several cuts simultaneously, e.g., making use of Gentzen's "mix" rule for GCL,

$$\frac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma'_\alpha \Rightarrow \Delta', \Delta_\alpha}$$

where Γ' has at least one occurrence of α, and Γ'_α and Δ_α are obtained by removing all occurrences of α from Γ' and Δ, respectively.

Cut elimination: Applications

- Consistency

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- Consistency
- Disjunction property

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- Decidability

Cut elimination: Applications

- Consistency
- Disjunction property
- Decidability
- Decidability of the equational theories for corresponding varieties of algebras

Substructural logics (1)

Consider the rules

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta} \ (\wedge\Rightarrow)' \qquad \frac{\Gamma \Rightarrow \Delta_1, \alpha, \beta, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha \vee \beta, \Delta_2} \ (\Rightarrow\vee)'$$

$$\frac{\Gamma_1, \alpha \Rightarrow \Delta_1 \quad \Gamma_2, \beta \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \alpha \vee \beta \Rightarrow \Delta_1, \Delta_2} \ (\vee\Rightarrow)' \qquad \frac{\Gamma_1 \Rightarrow \alpha, \Delta_1 \quad \Gamma_2 \Rightarrow \beta, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \wedge \beta, \Delta_1, \Delta_2} \ (\Rightarrow\wedge)'$$

It is an easy exercise to see that these rules are interderivable with the previous rules given for $\wedge$ and $\vee$, *making crucial use of the structural rules of weakening, exchange, and contraction*.
In the absence of such rules, the connectives $\wedge$ and $\vee$ split into two. That is, the original rules define what are often called the *additive* or *lattice* connectives $\wedge$ and $\vee$, whereas the alternatives rules define the so-called *multiplicative* or *group* connectives, renamed $\cdot$ and $+$.
Moreover, in the absence of weakening rules, the constants 1 and 0 also split, as in the absence of exchange rules, does the implication connective $\rightarrow$

Substructural logics (2)

The expression "substructural logic" was suggested by K. Došen and P. Schroeder-Heister to describe a family of logics emerging with a wide range of motivations from linguistics, algebra, set theory, philosophy, and computer science.
Roughly speaking, the term "substructural" refers to the fact that these logics *fail to admit one or more classically sound structural rules*. Observe that further logical rules may be added to capture connectives that split when structural rules are removed, and that "weaker" versions of the missing structural rules may also be added, giving a family of logics characterized by cut-free sequent calculi.

Nevertheless, there remain important classes of logics (e.g., relevant and fuzzy logics) typically accepted as substructural that do not fit comfortably into this framework, requiring more flexible formalisms such as hypersequents, display calculi, etc. More perplexing still, there are closely related logics for which no reasonable cut-free calculus is known. Are these also substructural?

Substructural logics (3)

A practical answer (Galatos et al., 2007): define substructural logics by appeal to their algebraic semantics. That is, since most substructural logics correspond in some way to classes of *residuated lattices*, this family could be identified with logics having these classes of algebras as equivalent algebraic semantics.

Such a definition offers uniformity and clarity, although there are problems with:

- fragments;
- extensions;
- non-algebraizable logics;
- ...

Lambek calculus (1)

J. Lambek (1958) made use of a substructural sequent calculus to represent transformations on syntactic types of a formal grammar. Lambek's approach built on earlier work on *categorial grammar* (Ajdukiewicz, 1930's), aimed at developing an analysis of natural language by assigning syntactic types to linguistic expressions that describe their syntactic roles (e.g., verb, noun phrase, verb phrase, sentence).

A naive approach to this task would consist of listing a number of lexical atoms (e.g., **Joan**, **smiles**, **charmingly**) and a number of mutually unrelated types (e.g., NP = noun phrase; V = verb; Adv = adverb; VP = verb phrase; S = sentence), and then tagging each lexical atom with the appropriate type:

Joan: NP; **smiles**: V; **charmingly**: Adv

Lambek calculus (2)

Ajdukiewicz suggested that the stock of basic types can be substantially reduced by resorting to the *type-forming operators* $\backslash$ and $/$. An expression v has type $\alpha\backslash\beta$ (respectively, β/α) if, whenever v' has type α, the expression $v'v$ (respectively, vv') has type β. This way, categorial grammar can be constructed out of just two types: n (noun) and s (sentence).

Example: in English, **John works** is a sentence, **works John** is not. **works** has type $n\backslash s$: when applied to the right of an expression of type n, it yields an expression of type s. Similarly, **poor** has type n/n. We may write these transformations as $n, n\backslash s \Rightarrow s$ and $n, n/n \Rightarrow n$. More generally, the following transformations are permissible in categorial grammar:

$$\alpha, \alpha\backslash\beta \Rightarrow \beta \text{ and } \beta/\alpha, \alpha \Rightarrow \beta.$$

Lambek calculus (3)

Lambek extended the deductive power of categorial grammar by setting up a sequent calculus for permissible transformations on types, introducing a new type-forming operation $\cdot$ such that v has type $\alpha \cdot \beta$ whenever $v = v'v''$ with v' of type α and v'' of type β, and admitting, in addition to modus ponens, patterns of hypothetical reasoning corresponding to right introduction rules for the implications.

Adding rules for the lattice connectives $\wedge$ and $\vee$, and the constants 1 and 0, gives the *Full Lambek Calculus* GFL.

The Full Lambek Calculus GFL (1)

Axioms

$$\frac{}{\alpha \Rightarrow \alpha}\ (\text{ID})$$

Cut rule

$$\frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2, \alpha, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\text{CUT})$$

Left logical rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Delta}\ (1 \Rightarrow)$$

$$\frac{}{0 \Rightarrow}\ (0 \Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \beta/\alpha, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (/ \Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \alpha\backslash\beta, \Gamma_3 \Rightarrow \Delta}\ (\backslash \Rightarrow)$$

Right logical rules

$$\frac{}{\Rightarrow 1}\ (\Rightarrow 1)$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash)$$

The Full Lambek Calculus GFL (2)

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_2 \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Delta}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

The bridge with residuated lattices (1)

We expect that the variety $\mathcal{FL}$ of FL-algebras be, somehow, the 'algebraic counterpart' of GFL. What do we mean by that?
For classical logic and Boolean algebras, this was easy to explain. We had a Hilbert-style calculus HCL, and a propositional logic $\mathrm{CL} = (\mathbf{Fm}, \vdash_{\mathrm{HCL}})$ obtained therefrom according to a standard procedure. We just showed that CL is algebraizable with $\mathcal{BA}$ as equivalent algebraic semantics.
Here, we have no Hilbert-style calculus (there is indeed one, but is rather unwieldy), just a sequent calculus. How can we get a logic from this calculus?

The bridge with residuated lattices (2)

We already have a formula algebra **Fm**, the formula algebra of the language of GFL. We need a consequence relation. We define, for $\Gamma \cup \{\alpha\} \subseteq Fm$:

$$\Gamma \vdash_{\mathrm{GFL}} \alpha \qquad \text{iff } \Rightarrow \alpha \text{ is derivable in the calculus obtained from GFL adding } \Rightarrow \gamma \text{ as initial axiom, for all } \gamma \in \Gamma.$$

$\vdash_{\mathrm{GFL}}$ is a consequence relation, so $\mathrm{FL} = (\mathbf{Fm}, \vdash_{\mathrm{GFL}})$ is a propositional logic.

The bridge with residuated lattices (3)

Theorem

FL *is algebraizable with* $\mathcal{FL}$ *as equivalent algebraic semantics.*

Proof.

Let

$$\begin{aligned} \tau(\alpha) &= \{\alpha \wedge 1 \approx 1\} \\ \rho(\alpha, \beta) &= \{\alpha \backslash \beta, \beta \backslash \alpha\}. \end{aligned}$$

We have to show:

1. $\Gamma \vdash_{\mathrm{FL}} \alpha$ iff $\tau(\Gamma) \vdash_{Eq(\mathcal{FL})} \tau(\alpha)$;
2. $\alpha \approx \beta \dashv\vdash_{Eq(\mathcal{FL})} \tau(\rho(\alpha, \beta))$.

□

The bridge with residuated lattices (4)

Proof.

(1), left to right: induction on the length of the derivation of α in $\mathrm{GFL} + \{\Rightarrow \gamma : \gamma \in \Gamma\}$.
(1), right to left: for $\Sigma \cup \{(\alpha, \beta)\} \subseteq Fm^2$, let

$$\Sigma^{\Rightarrow} = \left\{\alpha' \Rightarrow \beta' : \left(\alpha', \beta'\right) \in \Sigma\right\}; \Sigma^{\leq} = \left\{\alpha' \leq \beta' : \left(\alpha', \beta'\right) \in \Sigma\right\}.$$

We prove that

$$\Sigma^{\leq} \vdash_{Eq(\mathcal{FL})} \alpha \leq \beta \text{ implies } \Sigma^{\Rightarrow} \vdash_{\mathrm{GFL}} \alpha \Rightarrow \beta.$$

The result follows swiftly by taking $\alpha = \alpha' = 1$. □

The bridge with residuated lattices (5)

Proof.

The relation

$$\alpha\Theta_\Sigma\beta \text{ iff } \Sigma^{\Rightarrow} \vdash_{\mathrm{GFL}} \alpha \Rightarrow \beta \text{ and } \Sigma^{\Rightarrow} \vdash_{\mathrm{GFL}} \beta \Rightarrow \alpha$$

is a congruence on **Fm** (prove it!). Moreover, $\mathbf{A} = \mathbf{Fm}/\Theta_\Sigma$ is an FL-algebra. Now, suppose $\Sigma^{\Rightarrow} \nvdash_{\mathrm{GFL}} \alpha \Rightarrow \beta$; if we evaluate each p in $\Sigma \cup \{(\alpha, \beta)\}$ as its own congruence class modulo Θ_Σ, then $\alpha'^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_\Sigma}}) \leq \beta'^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_\Sigma}})$ for all $(\alpha', \beta') \in \Sigma$ (since each such $\alpha' \Rightarrow \beta'$ belongs to $\Sigma^{\Rightarrow}$), yet $\alpha^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_\Sigma}}) \not\leq \beta^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_\Sigma}})$ (since $\Sigma^{\Rightarrow} \nvdash_{\mathrm{GFL}} \alpha \Rightarrow \beta$).
(2) Observe that $\tau(\rho(\alpha, \beta)) = \{1 \leq \alpha\backslash\beta, 1 \leq \beta\backslash\alpha\}$and that $1 \leq \alpha\backslash\beta \dashv\vdash_{Eq(\mathcal{FL})} \alpha \leq \beta$. □

FL and its extensions

FL has come to play a distinguished role in the field of substructural logics. Just as classical logic is a candidate for the top element of the lattice of such logics, so FL is a candidate for the bottom element. That is, most other substructural logics may be obtained as extensions of FL.

In particular, H. Ono and colleagues have popularized the usage of FL_X where $X \subseteq \{e, c, w\}$ to denote the logic arising from GFL extended with the appropriate grouping of exchange (e), contraction (c), and weakening rules (w) in the same way as FL is obtained from GFL, and InFL_X to denote the logic arising from the corresponding multiple-conclusion sequent calculus. In particular, FL_{ewc} and InFL_{ewc} correspond to IL and CL with split connectives.

Some observations

- Let GRL be the sequent calculus obtained from GFL by removing the rules for 0. The corresponding logic RL is algebraizable with $\mathcal{RL}$ as equivalent algebraic semantics.
- The algebraizability results extends easily to FL_X-algebras and the logics FL_X for $X \subseteq \{e, c, w\}$.